On a theorem of Garza regarding algebraic numbers with real conjugates

Gerald Höhn*

March 23, 2010

1 Introduction. For an algebraic number α, that is, a root of an irreducible polynomial $\phi(x)$ with integer coefficients, the absolute height of α is defined by $H(\alpha) = |c|^{1/d}\prod_{i=1}^{d}\max(1,\,|\alpha_i|)^{1/d}$ in case $\phi(x) = c\prod_{i=1}^{d}(x-\alpha_i)$. The following lower estimate for the absolute height of α was recently found by J. Garza ([G], Theorem 1):

Theorem: *Let $\alpha \neq 0, \pm 1$ be an algebraic number with $r > 0$ real Galois conjugates. Then*

$$H(\alpha) \geq \left(\frac{2^{1-1/R} + \sqrt{4^{1-1/R}+4}}{2}\right)^{R/2}$$

where $R = r/d$ is the fraction of Galois conjugates α_i of α which are real.

If $R = 1$, i.e., α is a totally real, the bound simplifies to Schinzel's estimate (see [S], Corollary 1')

$$H(\alpha) \geq \left(\frac{1+\sqrt{5}}{2}\right)^{1/2}$$

stated in loc. cit. for algebraic integers only. A short proof of Schinzel's bound in this case was given in [HS]. In this note we show that a similar method as in [HS] together with basic properties of absolute values of number fields also leads to a new derivation of Garza's bound.

2 Proof of Theorem. We start with an elementary estimate.

Lemma: *For $0 < a < \frac{1}{2}$ let $f(x) = |x|^{1/2-a}|1-x^2|^a$. Then the function $f(x)/\max(1,\,|x|)$ has the global maximum $M_{\mathbf{C}} = 2^a$ on the complex plane and the global maximum*

$$M_{\mathbf{R}} = (4a)^a(1-2a)^{1/4-a/2}(1+2a)^{-1/4-a/2}$$

on the real axis.

*Department of Mathematics, Kansas State University, 138 Cardwell Hall, Manhattan, KS 66506-2602, USA. E-mail: gerald@math.ksu.edu

Proof of the lemma: One has $f(x) \le 2^a$ for $|x| \le 1$ and $f(i) = 2^a$. For $|x| \ge 1$ one gets $f(x)/|x| \le |x|^{-1/2-a}(2|x|^2)^a \le 2^a$ proving the first statement. For the second statement, one verifies by using the first derivative and computing the boundary values that $f(x)$ reaches the stated global maximum in the interval $[0,1]$ at $x_1 = \sqrt{\frac{1-2a}{1+2a}}$ and that $f(x)/|x|$ reaches the same global maximum in the interval $[1,\infty)$ at $x_2 = \sqrt{\frac{1+2a}{1-2a}}$. $\square$

Continuing with the notation from the lemma, one has for an algebraic integer α the estimate

$$\prod_{i=1}^{d} f(\alpha_i) = |\phi(0)|^{1/2-a}|\phi(1)\phi(-1)|^a \ge 1.$$

Therefore,

$$\prod_{i=1}^{d} \max(1,\ |\alpha_i|) \ge M_{\mathbf{R}}^{-r} M_{\mathbf{C}}^{r-d} \prod_{i=1}^{d} f(\alpha_i) \ge M_{\mathbf{R}}^{-r} M_{\mathbf{C}}^{r-d}$$

or $H(\alpha) \ge M_{\mathbf{R}}^{-R} M_{\mathbf{C}}^{R-1}$ for the height. Applying the lemma for $a = \frac{1}{2}(1+4^{1/R})^{-1/2}$ gives

$$\begin{aligned} H(\alpha) &\ge (4a)^{-aR}(1-2a)^{(a/2-1/4)R}(1+2a)^{(a/2+1/4)R}2^{a(R-1)} \\ &= \left(\left(\frac{1+4^{1/R}}{4}\right)^a \left(\frac{4^{1/R}}{1+4^{1/R}}\right)^a 4^{a(1-1/R)} \cdot \frac{1+2a}{(1-4a^2)^{1/2}}\right)^{R/2} \\ &= \left(\left(\frac{1+4^{1/R}}{4^{1/R}}\right)^{1/2} \left(1+(1+4^{1/R})^{-1/2}\right)\right)^{R/2} = \left(\frac{2^{1-1/R}+\sqrt{4^{1-1/R}+4}}{2}\right)^{R/2}, \end{aligned}$$

which finishes the proof of the theorem in the case of the algebraic integers.

The above argument can be extended to abitrary algebraic numbers α by using some basic algebraic number theory and properties of the absolute height (cf. [I] for the case of Schinzel's result).

Let $k = \mathbf{Q}(\alpha)$. For a place ν of k we denote by $|\,.\,|_\nu$ the corresponding normalized absolute value of k, so that $\prod_v |\beta|_\nu = 1$ for a non-zero algebraic number β in k. Then the absolute height of β equals $H(\beta) = \prod_v \max(1, |\beta|_\nu)$. With $a \le 1/2$ as above, we have the estimate

$$\begin{aligned} 1 &= \prod_{\nu} |\alpha - \alpha^{-1}|_\nu^a = \prod_{\nu|\infty} |\alpha - \alpha^{-1}|_\nu^a \cdot \prod_{\nu\nmid\infty} |\alpha - \alpha^{-1}|_\nu^a \\ &\le \prod_{\nu|\infty} |\alpha - \alpha^{-1}|_\nu^a \prod_{\nu\nmid\infty} \max(1, |\alpha|_\nu)^a \max(1, |\alpha^{-1}|_\nu)^a \\ &\le \prod_{\nu|\infty} \frac{(|\alpha_\nu - \alpha_\nu{}^{-1}|^a)^{d_\nu/d}}{(\max(1,|\alpha_\nu|)^{1/2}\max(1,|\alpha_\nu^{-1}|)^{1/2})^{d_\nu/d}} \cdot \prod_{\nu} \max(1,|\alpha|_\nu)^{1/2}\max(1,|\alpha|_\nu^{-1})^{1/2} \end{aligned}$$

where $d_\nu = [\,k_\nu : \mathbf{R}]$ and α_ν is the image of α under some Galois automorphism of the Galois closure of k such that $|\alpha|_\nu = |\alpha_\nu|^{d_\nu/d} = |\alpha_i|^{d_\nu/d}$ for some i so that one factor for

each pair $\{\alpha_i, \bar{\alpha}_i\}$ appears in the product over the archimedean places. Since $g(x) = |x - x^{-1}|^a/(\max(1,|x|)^{1/2} \max(1,|x^{-1}|)^{1/2})$ is symmetric under $x \mapsto x^{-1}$ we can assume $|x| \geq 1$ where $g(x) = f(x)/\max(1,|x|)$. By applying the lemma we get now the estimate

$$1 \leq M_{\mathbf{R}}^R M_{\mathbf{C}}^{1-R} \cdot H(\alpha)^{1/2} H(\alpha^{-1})^{1/2}$$

and the result follows as before by using $H(\alpha) = H(\alpha^{-1})$.

3 Remarks. 1. Under all functions $\tilde{f}(x) = |x|^u|1-x^2|^v$, the chosen $f(x)$ gives the best estimate for $H(\alpha)$.

2. For $R = 1$ the bound for $H(\alpha)$ is optimal. One may ask if this is also the case for other values of R, although it follows from the proof that there cannot exist an α actually reaching the bound.

3. The main difference to Garza's proof is that we replace a sequence of inequalities in [G] with the estimate of the lemma, allowing a particular elementary proof for algebraic integers.

References

[G] J. Garza, *On the height of algebraic numbers with real conjugates,* Acta Arith. **128** (2007), 385–389.

[HS] G. Höhn and N.-P. Skoruppa, *Un résultat de Schinzel,* J. Théor. Nombres Bordeaux, **5** (1993), 185.

[I] M. I. M. Ishak, *Lower bounds for heights in cyclotomic extensions and related problems,* Ph.D. thesis, Kansas State University (2009).

[S] A. Schinzel, *On the product of the conjugates outside the unit circle of an algebraic number,* Acta Arith. **24** (1973), 385–399; Addendum, Acta Arith. **26** (1973), 329–331.